SOME OPEN QUESTIONS ON A RESULT OF B.H. NEUMANN

Francesco G. Russo

Abstract. A subgroup $K$ of a group $G$ is called almost normal in $G$ if it has finitely many conjugates in $G$. The influence of these subgroups is strong on the group structure. Indeed, B.H. Neumann proves in the 1955 that $|G : Z(G)|$ is finite if and only if each $K$ is almost normal in $G$. Many authors have successively generalized this result and the present survey makes the point of the situation, illustrating a new perspective for wider generalizations.

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1. A brief overview

In [1] R. Baer describes the structure of the groups with finite conjugacy classes, or $FC$-groups. Some years later B.H. Neumann writes the two papers [34] and [35] which will be classic works in the theory of $FC$-groups. See also [41, Vol.I, §4.3]. It is a common opinion that [1, 34, 35] introduce a new approach of study of the infinite groups. Some results, which originated from [1, 34, 35], are in [2, 7, 8, 9, 10, 14, 17, 18, 19, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49]. The list is very partial and reflects only some topics which we will illustrate successively.

The investigations in [34] and [35] differ from those in [1] for two main motivations. The first deals with the bounds of the finite conjugacy classes (see [41, Theorem 4.35] or [34, Theorem 3.1]). The second deals with the covering properties of a group by means of suitable subgroups (see [41, Theorem 4.16, Lemma 4.17]). More precisely, [41, Theorem 4.35] states that a group $G$ has finite index $|G : Z(G)|$ if and only if the conjugacy classes of $G$ are finite and bounded, or equivalently, if and only if $G'$ is finite. [41, Theorem 4.16] states that $|G : Z(G)|$ is finite if and only if $G$ has a finite covering consisting of abelian subgroups. It is clear the connection among the theory of the coverings of subgroups and that of...
FC-groups. Variations on these themes have interested many authors in different contexts in the last years. See for instance [5, 6, 7, 8, 9, 10, 17, 18, 19].

2. Anti-XC-groups and Neumann’s Theorem

Assume from now that \( \mathcal{X} \) denotes an arbitrary class of groups which is closed with respect to forming subgroups and quotients, \( \mathfrak{F} \) is the class of all finite groups, \( \mathfrak{F}_\pi \) is the class of all finite \( \pi \)-groups (\( \pi \) set of primes), \( \mathfrak{C} \) is the class of all Chernikov groups, \( \mathfrak{P} \mathfrak{F} \) is the class of all polycyclic-by-finite groups, \( \mathfrak{S}_2 \mathfrak{F} \) is the class of all (soluble minimax)-by-finite groups. It is easy to check that

\[
\mathfrak{F} \subseteq \mathfrak{C} \subseteq \mathfrak{S}_2 \mathfrak{F}, \quad \mathfrak{F} \subseteq \mathfrak{P} \mathfrak{F} \subseteq \mathfrak{S}_2 \mathfrak{F}, \quad \mathfrak{C} \cap \mathfrak{P} \mathfrak{F} = \mathfrak{F}.
\]

See [30, 31, 32, 41, 42] for details.

Given a positive integer \( r \) and a group \( G \), we recall that the operator \( L \), defined by

\[
L \mathcal{X} = \{ G \mid \langle g_1, g_2, \ldots, g_r \rangle \in \mathcal{X}, \forall g_1, g_2, \ldots, g_r \in G \},
\]

from \( \mathcal{X} \) to \( \mathcal{X} \) is called local operator for \( \mathcal{X} \). See [31, \( \mathcal{C} \), p.54]. We recall that the operator \( H \), which associates to \( \mathcal{X} \) the class of hyper-\( \mathcal{X} \)-groups is called extension operator. See [31, \( \mathcal{E} \), p.60]. Notations and terminology follow [30, 31, 32, 41, 42].

As already recalled in the abstract, a subgroup \( K \) of a group \( G \) is called almost normal in \( G \) if \( K \) has finitely many conjugates in \( G \), that is, if \( |G:N_G(K)| \) is finite. Neumann’s Theorem [41, Chapter 4, Vol.I, p.127] shows that \( G \) has each \( K \) which is almost normal in \( G \) if and only if \( G/Z(G) \in \mathfrak{F} \). We have

\[
N_G(Cl_G(K)) = \text{core}_G(N_G(K)) = \bigcap_{x \in G} N_G(K)^x = \bigcap_{x \in G} N_G(K^x),
\]

where \( Cl_G(K) \) is the set of conjugates of \( K \) in \( G \). \( |G:N_G(K)| = |Cl_G(K)| \) is finite if and only if \( G/\text{core}_G(N_G(K)) \in \mathfrak{F} \). In [25, 26] \( G \) has \( \mathfrak{F} \)-classes of conjugate subgroups, if \( G/\text{core}_G(N_G(K)) \in \mathfrak{F} \) for each \( K \) in \( G \).

Thus Neumann’s Theorem can be reformulated as follows.

**Theorem 2.1 (Neumann’s Theorem).** A group \( G \) has \( \mathfrak{F} \)-classes of conjugate subgroups if and only if \( G/Z(G) \in \mathfrak{F} \).

See [26, Introduction]. More generally, \( G \) has \( \mathcal{X} \)-classes of conjugate subgroups, if \( G/\text{core}_G(N_G(K)) \in \mathcal{X} \) for each \( K \) in \( G \). In this context there are two questions of great interest.

**Open Question 2.2.** For which choice of \( \mathcal{X} \), in a group \( G \) the condition to have \( \mathcal{X} \)-classes of conjugate subgroups is equivalent to \( G/Z(G) \in \mathcal{X} \)?

Theorem 2.1 answers positively Question 2.2 for \( \mathcal{X} = \mathfrak{F} \). We know a positive answer of Question 2.2 also for \( \mathcal{X} = \mathfrak{P} \mathfrak{F} \) from [25, Main Theorem]. Indeed, this result states that a group \( G \) has \( \mathfrak{P} \mathfrak{F} \)-classes of conjugate subgroups if and only if \( G/Z(G) \in \mathfrak{P} \mathfrak{F} \). Unfortunately, Question 2.2 has a negative answer for \( \mathcal{X} = \mathfrak{C} \). [26, Main Theorem] describes groups having \( \mathfrak{C} \)-classes of conjugate subgroups and [26,
Section 4] shows an example of a group having $C$-classes of conjugate subgroups with $G/Z(G) \not\in \mathcal{C}$. Therefore Question 2.2 should be strengthened as follows.

Open Question 2.3. What is the structure of a group having $X$-classes of conjugate subgroups?

Question 2.3 is partially answered in [44, Main Theorem], where there is a description of the groups having $S_2\mathcal{F}$-classes of conjugate subgroups. Here some restrictions are done and so Question 2.2 could be answered positively. This is an open problem.

Recall that $Z_X(G) = \{x \in G \mid G/C_G(\langle x \rangle^G) \in \mathcal{X}\}$ is a characteristic subgroup of $G$, called $XC$-center of $G$. See [31, Definition B.1, Proposition B.2]. $G$ is called $XC$-group if it coincides with its $XC$-center. $FC$-groups, $CC$-groups, $PC$-groups and $MC$-groups are obtained when we consider respectively $\mathcal{F}$, $\mathcal{C}$, $\mathcal{PS}$, $\mathcal{S}_2\mathcal{F}$. These are studied in [2, 14, 28, 30, 31, 32, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47]. Finally, $G$ is called $HXC$-group if $G = Z_{HX}(G)$.

If $G$ has $\mathcal{F}$-classes of conjugate subgroups, then it is an $FC$-group. From [26, Lemma 2.3], if $G$ has $\mathcal{C}$-classes of conjugate subgroups, then it is a $CC$-group. From [25, Corollary 2.7], if $G$ has $\mathcal{PS}$-classes of conjugate subgroups, then it is a $PC$-group. From [44, Lemma 2.4], if $G$ has $\mathcal{S}_2\mathcal{F}$-classes of conjugate subgroups, then it is an $MC$-group. These facts can be generalized in the next form.

Lemma 2.4. ([46, Lemma 2.1]) Assume that $\mathcal{F}_X = \mathcal{X}$. If $G$ has $X$-classes of conjugate subgroups, then $Z_X(G) = G$.

We recall that $\mathcal{X}$ is called Dietzmann class, if for every group $G$ and $x \in G$, the following implication is true:

\[(*) \quad \text{if} \ x \in Z_X(G) \text{ and } (x) \in \mathcal{X}, \text{ then } \langle x \rangle^G \in \mathcal{X}.\]

See [31, Definitions B.1 and B.6] or [11]. Dietzmann classes are studied in [30, 31, 32, 42]. $FC$-groups form a Dietzmann class as we note in [31, Proposition D.3, b)]. In particular, this is true for periodic $PC$-groups, which are obviously $FC$-groups. Note that $\mathcal{F}$ is a Dietzmann class (see [31, Proposition B.7, b)] but $\mathcal{P}\mathcal{S}$ is not a Dietzmann class (see [31, Example B.8, c)]). Unfortunately, it is not known whether $PC$-groups, $CC$-groups or $MC$-groups form a Dietzmann class. See always [30, 31, 32, 42]. But it is easy to check that $PC$-groups, $CC$-groups or $MC$-groups extend locally the class of $FC$-groups. Therefore, the next result is significant.

Theorem 2.5. ([31, Theorem E.3]) If $\mathcal{F}_\pi \subseteq \mathcal{X} \subseteq \mathcal{L}\mathcal{F}_\pi$, then the $HXC$-groups form a Dietzmann class.

From Lemma 2.4 and Theorem 2.5, it is meaningful to ask whether we may strengthen Neumann’s Theorem, considering the following property:

\[(**) \quad \text{if} \ K \text{ is a non-finitely generated subgroup of a group } G, \text{ then } G/core_G(N_G(K)) \in \mathcal{X}, \text{ where } \mathcal{F}_\pi \subseteq \mathcal{X} \subseteq \mathcal{L}\mathcal{F}_\pi.\]
$G$ is called anti-XC-group if it satisfies (**) They are studied in [46] Anti-FC-groups are described in [12] Anti-CC-groups and anti-PC-groups are described in [45] We cannot forget in this line of research [20] whose methods are used both in [12] and [20] On another hand, the ideas and the methods go back to [33] and deal with the structure of groups with given properties of a system of subgroups Among the impressive literature in this topic, we mention [3, 4, 13, 15, 21, 22, 23, 24, 27, 29, 37, 51].

3. Locally finite case

Following [12, 20, 45], in this Section we will give a brief description of the locally finite groups satisfying (**) They are discussed in [46] The considerations in Section 2 allow us to prove easily the next two results.

Lemma 3.1. Subgroups and quotients of anti-XC-groups are anti-XC-groups.

Lemma 3.2. If $G$ is an anti-XC-group and $Z_X(G) = G$, then $G$ has $X$-classes of conjugate subgroups.

Overlapping [45, Lemma 3.3] and from Lemmas 3.1 and 3.2, we have as follows.

Lemma 3.3. Assume that $x$ is an element of the anti-XC-group $G$. If $A = Dr_{i \in I} A_i$ is a subgroup of $G$ consisting of ($x$)-invariant nontrivial direct factors $A_i$, $i \in I$, with infinite index set $I$, then $x$ belongs to $Z_X(G)$.

Lemma 3.3 has the next consequence, which is straightforward.

Corollary 3.4. Assume that $G$ is an anti-XC-group and $A = Dr_{i \in I} A_i$ is a subgroup of $G$ consisting of infinitely many nontrivial direct factors. Then $A$ is contained in $Z_X(G)$.

The next lemma overlaps [45, Lemma 3.7].

Lemma 3.5. Assume that $g$ is an element of the anti-XC-group $G$ and $A = Dr_{i \in I} A_i$ is a subgroup of $G$, with $I$ as in Lemma 2.3. If $g \in N_G(A)$ and $g^n \in C_G(A)$ for some positive integer $n$, then $g$ belongs to $Z_X(G)$.

Combining the above Lemmas 3.1, 3.2, 3.3, 3.5 and Corollary 3.4 we get the next corollary, whose proof overlaps [45, Corollary 3.9].

Corollary 3.6. If the anti-XC-group $G$ has an abelian torsion subgroup that does not satisfy the minimal condition on its subgroups, then all elements of finite order belong to $Z_X(G)$.

All the above considerations allow us to describe the locally finite case.

Theorem 3.7. If $G$ is a locally finite anti-XC-group, then either $G$ has $X$-classes of conjugate subgroups or $G$ is a Chernikov group.
Proof. From Lemmas 3.1, 3.2, 3.3, 3.5 and Corollaries 3.4, 3.6, we may argue as in [45, Theorem 3.12, Proof], considering $X, Z_X(G)$ and the result follows. 

Theorem 3.7 extends [45, Theorems 3.11 and 3.12] and similar situations in [12, 20]. The locally nilpotent case can be treated in an analogous way, invoking some results in [50].

References


Francesco G. Russo
Department of Mathematics, University of Palermo, via Archirafi 34, Palermo, Italy
e-mail: francescog.russo@yahoo.com